

POWER OPERATIONS FOR MORAVA E -THEORY OF HEIGHT 2 AT THE PRIME 2

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ABSTRACT. Explicit calculations of the algebraic theory of power operations for a specific Morava E -theory spectrum are given, without detailed proofs.

1. INTRODUCTION

In this note, we give an explicit description of the algebraic theory of power operations for a particular Morava E -theory spectrum associated to the deformations of a supersingular elliptic curve at the prime 2. To understand the context behind the calculations, this note should be read together with [Rez]. The purpose of this note is to lay out some calculations (§2), without giving complete proofs, but merely indicating how they arise from algebraic geometry and topology (§3 and §4).

Warning: there is difference of convention with [Rez]. In that paper, it was convenient to arrange things so that the interesting objects were *right* Γ -modules. Here, because I'm more comfortable carrying out computations this way, we arrange things so that the interesting objects are *left* Γ -modules. To translate between the two conventions, replace Γ with Γ^{op} .

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2. ALGEBRAIC CONSTRUCTIONS

In this section, we will work with the ground ring $R = \mathbb{Z}[a]$, a polynomial ring on one generator.

2.1. The ring Γ . We define an associative ring Γ equipped with a ring homomorphism $\eta: R \rightarrow \Gamma$ as follows. The ring Γ is generated over R by elements Q_0 , Q_1 , and Q_2 , subject to (i) *commutation relations*, and (ii) *adem relations*. The commutation relations state that the Q_i 's commute with elements of $\mathbb{Z} \subset R$, and that

$$\begin{aligned} Q_0 a &= a^2 Q_0 - 2a Q_1 + 6 Q_2, \\ Q_1 a &= 3 Q_0 + a Q_2, \\ Q_2 a &= -a Q_0 + 3 Q_1. \end{aligned}$$

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The adem relations are

$$\begin{aligned} Q_1 Q_0 &= 2 Q_2 Q_1 - 2 Q_0 Q_2, \\ Q_2 Q_0 &= Q_0 Q_1 + a Q_0 Q_2 - 2 Q_1 Q_2. \end{aligned}$$

It follows using these relations that Γ has an *admissible basis*; that is, Γ is free as a left R -module on the elements of the form

$$Q_0^j Q_{k_1} \cdots Q_{k_r}, \quad j \geq 0, k_i \in \{1, 2\}, r \geq 0.$$

Note that if we write $\Gamma[k]$ for the degree k part of Γ , then $\text{rank } \Gamma[k] = 1 + 2 + \cdots + 2^k$.

2.2. Γ -modules. By Γ -module, we will mean a *left* Γ -module, unless otherwise specified. Any Γ -module is automatically an R -module, via the ring homomorphism $\eta: R \rightarrow \Gamma$. Given an R -module M , we say that a Γ -module structure on M *extends* the given R -module structure if these module structures are compatible with respect to η .

When there is possibility of confusion, for M a Γ -module, $\gamma \in \Gamma$, $m \in M$, we write $\gamma \cdot m \in M$ for the action of γ on m . Otherwise, we just write γm for this element.

There is a standard Γ -module structure on R , which is characterized as the unique one extending the R -module structure on R , and such that

$$\begin{aligned} Q_0 \cdot 1 &= 1, \\ Q_1 \cdot 1 &= 0, \\ Q_2 \cdot 1 &= 0. \end{aligned}$$

Given Γ -modules M and N , we define their tensor product $M \otimes N$ as follows. The underlying R -module is $M \otimes_R N$. The Γ -module structure is given by

$$\begin{aligned} Q_0(x \otimes y) &= Q_0 x \otimes Q_0 y + 2 Q_1 x \otimes Q_2 y + 2 Q_2 x \otimes Q_1 y, \\ Q_1(x \otimes y) &= Q_0 x \otimes Q_1 y + Q_1 x \otimes Q_0 y + a Q_1 x \otimes Q_2 y + a Q_2 x \otimes Q_1 y + 2 Q_2 x \otimes Q_2 y, \\ Q_2(x \otimes y) &= Q_0 x \otimes Q_2 y + Q_2 x \otimes Q_0 y + Q_1 x \otimes Q_1 y + a Q_2 x \otimes Q_2 y. \end{aligned}$$

The category of Γ -modules, equipped with the tensor product \otimes and unit object R , is a symmetric monoidal category.

2.3. The element Ψ . Let $\Psi \in \Gamma$ be the element defined by

$$\Psi = Q_0 Q_0 + a Q_0 Q_1 - 2 Q_1 Q_1 + a^2 Q_0 Q_2 - 2a Q_1 Q_2 + 4 Q_2 Q_2.$$

It is “easy” to check that Ψ is in the center of Γ , and that for a tensor product $M \otimes N$ of Γ -modules we have $\Psi(x \otimes y) = \Psi x \otimes \Psi y$.

2.4. The module ω . The Γ -module ω is defined as follows. The underlying R -module of ω is a free R -module on one generator u . The Γ -module structure is the unique one extending this R -module structure, and with

$$\begin{aligned} Q_0 \cdot u &= 0, \\ Q_1 \cdot u &= -u, \\ Q_2 \cdot u &= 0. \end{aligned}$$

Observe that $\Psi \cdot u = -2u$.

We write ω^n for the n th tensor power of ω , $n \geq 0$.

2.5. Γ -rings. A Γ -ring is a commutative monoid object in Γ -modules. Equivalently, a Γ -ring is a commutative R -algebra equipped with a Γ -module structure compatible with the R -module structure, and which satisfies the *cartan formulas*

$$\begin{aligned} Q_0(xy) &= Q_0x Q_0y + 2 Q_1x Q_2y + 2 Q_2x Q_1y, \\ Q_1(xy) &= Q_0x Q_1y + Q_1x Q_0y + a Q_1x Q_2y + a Q_2x Q_1y + 2 Q_2x Q_2y, \\ Q_2(xy) &= Q_0x Q_2y + Q_2x Q_0y + Q_1x Q_1y + a Q_2x Q_2y. \end{aligned}$$

The ring R , with its standard Γ -module structure, is a Γ -ring.

2.6. Frobenius congruence and amplified Γ -rings. We say that a Γ -ring A satisfies the *frobenius congruence* if for all $x \in A$, we have

$$Q_0x \equiv x^2 \pmod{2A}.$$

An *amplified Γ -ring* is a Γ -ring A , together with a function $\theta: A \rightarrow A$ which is a “witness” for the frobenius congruence. That is, the identity

$$Q_0x = x^2 + 2\theta x$$

holds for all $x \in A$, and furthermore a number of identities involving θ hold, all of which are identities necessarily satisfied in a torsion free Γ -ring by a function θ satisfying the above identity. Among these additional identities, we have

$$\begin{aligned} \theta(x+y) &= \theta x + \theta y - xy, \\ \theta(ax) &= a^2 \theta x - a Q_1x + 3 Q_2x, \\ \theta(xy) &= x^2 \theta y + y^2 \theta x + 2 \theta x \theta y + Q_1x Q_2y + Q_2x Q_1y, \\ Q_1\theta(x) &= Q_2Q_1x - Q_0Q_2x - Q_0x Q_1x - a Q_1x Q_2x - (Q_2x)^2, \\ Q_2\theta(x) &= \theta Q_1x + a \theta Q_2x - Q_1Q_2x - Q_0x Q_2x, \end{aligned}$$

for all $x, y \in A$. We also have $\Psi\theta = \theta\Psi$.

Amplified Γ -rings are an example of an *amplified plethory*, as in [BW05].

The free amplified Γ -ring on one generator x is a polynomial ring of the form

$$R[\theta^j Q_{k_1} \cdots Q_{k_r} x, j, r \geq 0, k_i \in \{1, 2\}].$$

2.7. Extension of scalars. Consider the R -algebras $S = \mathbb{Z}[a, (a^3 - 27)^{-1}]$ and $\hat{S} = \mathbb{Z}_2[[a]]$. These both admit in a unique way the structure of amplified Γ -ring. Note that this is not obviously the case, since a is not central in Γ . For instance, to show that the action of Γ can be defined on \hat{S} , one shows that $\Gamma \cdot 2R \subseteq 2R$ and $\Gamma \cdot a^3 R \subseteq 2R + aR$, and thus the elements of Γ act continuously on R with respect to the adic topology.

The quotient ring $R/(2)$ is a Γ -ring, but is not an amplified Γ -ring. The quotient ring $R/(2, a)$ is not even a Γ -ring.

2.8. Trace, norm, and logarithm operators. Define the element T in Γ by

$$T = 3Q_0 + 2aQ_2.$$

For a Γ -ring A , we define a function $N: A \rightarrow A$ by

$$\begin{aligned} Nx = (Q_0x)^3 + 2a(Q_0x)^2Q_2x - aQ_0x(Q_1x)^2 + a^2Q_0x(Q_2x)^2 \\ - 6Q_0Q_1xQ_2x + 2(Q_1x)^3 - 2aQ_1x(Q_2x)^2 + 4(Q_2x)^3. \end{aligned}$$

The function N is multiplicative: $N(xy) = (Nx)(Ny)$ for any $x, y \in A$.

For $m \in \mathbb{Z} \subset R$, we have $N(m) = m^3$. One can compute that

$$N(a-3) = -(a-3)^3, \quad N(a^3-27) = -(a^3-27)^3.$$

The operator T is the “linearization” of N . That is, if $A \rightarrow A/I$ is a homomorphism of Γ -rings, then for $x \in I$ we have

$$N(1+x) \equiv Tx \pmod{I^2}.$$

It is straightforward to show that if A is an amplified Γ -ring, and $x \in A$, then the congruence

$$Nx \equiv x^2 \Psi x \pmod{2A}$$

holds. Let $A^\times \subset A$ denote the group of units in A . The above congruence implies that there is a function $M: A^\times \rightarrow A$ defined by

$$1 + 2Mx = \frac{x^2 \Psi x}{Nx}.$$

Now define a homomorphism $\ell: A^\times \rightarrow A_2^\wedge$ by

$$\ell(x) = \frac{1}{2} \log \left(\frac{x^2 \Psi x}{Nx} \right) = \sum_{k \geq 1} (-1)^{k-1} \frac{2^{k-1}}{k} (Mx)^k.$$

As an example, observe that for $D = a^3 - 27$ in $S = \mathbb{Z}[a][D^{-1}]$, we have

$$\ell(D) = \frac{1}{2} \log \left(\frac{D^2 \Psi D}{ND} \right) = \frac{1}{2} \log(-1) = 0.$$

In fact, $\ell(x) = 0$ for all $x \in S^\times$. (This is not true for all elements of \widehat{S}^\times .)

2.9. Exercise. Calculate the kernel of $\ell: \widehat{S}^\times \rightarrow \widehat{S}$.

2.10. A Koszul complex for Γ . Let $C_1 = Rq_0 + Rq_1 + Rq_2$, a free left R -module on 3 generators q_0, q_1, q_2 . We impose a right R -module structure on C_1 (distinct from the left R -module structure), by

$$\begin{aligned} q_0 a &= a^2 q_0 - 2a q_1 + 6 q_2, \\ q_1 a &= 3 q_0 + a q_2, \\ q_2 a &= -a q_0 + 3 q_1. \end{aligned}$$

(Thus C_1 is evidently isomorphic to the degree 1 part $\Gamma[1] \subset \Gamma$, as a sub- R -bimodule of Γ .)

Let $C_2 = Rr_1 + Rr_2$, a free left R -module on 2 generators r_1, r_2 . We define a right R -module structure on C_2 to coincide with the left R -module structure, so that $r_1 a = ar_1$ and $r_2 a = ar_2$. (We can identify C_2 with the kernel of multiplication $\Gamma[1] \otimes_R \Gamma[1] \rightarrow \Gamma[2]$.)

Given a (left) Γ -module M , we define a chain complex

$$0 \rightarrow \Gamma \otimes_R C_2 \otimes_R M \xrightarrow{d_2} \Gamma \otimes_R C_1 \otimes_R M \xrightarrow{d_1} \Gamma \otimes_R M \xrightarrow{d_0} M \rightarrow 0$$

as follows. The tensor products are as R -bimodules. The differentials are defined by

$$\begin{aligned} d_0(\gamma \otimes m) &= \gamma m, \\ d_1(\gamma \otimes q_0 \otimes m) &= \gamma \otimes Q_0 m - \gamma Q_0 \otimes m, \\ d_1(\gamma \otimes q_1 \otimes m) &= \gamma \otimes Q_1 m - \gamma Q_1 \otimes m, \\ d_1(\gamma \otimes q_2 \otimes m) &= \gamma \otimes Q_2 m - \gamma Q_2 \otimes m, \\ d_2(\gamma \otimes r_1 \otimes m) &= \gamma Q_1 \otimes q_0 \otimes m + \gamma(-2Q_2) \otimes q_1 \otimes m + \gamma(2Q_0) \otimes q_2 \otimes m \\ &\quad + \gamma \otimes q_1 \otimes Q_0 m + \gamma \otimes (-2q_2) \otimes Q_1 m + \gamma \otimes (2q_0) \otimes Q_2 m, \\ d_2(\gamma \otimes r_2 \otimes m) &= \gamma Q_2 \otimes q_0 \otimes m + \gamma(-Q_0) \otimes q_1 \otimes m + \gamma(-aQ_0) \otimes q_2 \otimes m + \gamma(2Q_1) \otimes q_2 \otimes m \\ &\quad + \gamma \otimes q_2 \otimes Q_0 m + \gamma \otimes (-q_0) \otimes Q_1 m + \gamma \otimes (-aq_0) \otimes Q_2 m + \gamma \otimes (2q_1) \otimes Q_2 m. \end{aligned}$$

The ideas of [Pri70] apply to show that this complex of left Γ -modules is acyclic when M is flat as an R -module.

2.11. *Exercise.* Let $I \subset \Gamma$ denote the two-sided ideal generated by Q_0, Q_1, Q_2 . Regard the quotient Γ/I as a *right* Γ -module. Use the Koszul complex to calculate $\mathrm{Tor}_q^\Gamma(\Gamma/I, \omega^k)$ for various values of $k \geq 0$.

3. AN ELLIPTIC CURVE

Let $R = \mathbb{Z}[a]$. Consider the curve C over R defined by the projective equation

$$Y^2 Z + aXYZ + YZ^2 - X^3 = 0$$

in \mathbb{P}_R^3 . In terms of affine coordinates $x = X/Z$ and $y = Y/Z$, this is the Weierstrass curve with equation

$$y^2 + axy + y = x^3.$$

Over the extension ring $S = R[D^{-1}]$ with $D = a^3 - 27$, this is a (smooth) elliptic curve.

It is more convenient for us to consider an affine neighborhood about the point $O = [0 : 1 : 0] \in \mathbb{P}^3$. Thus, if we take $u = X/Y$ and $v = Z/Y$, our curve has the equation

$$v^2 + auv + v = u^3,$$

where $(u, v) = (0, 0)$ is the identity of the elliptic curve.

Let Q be a point on this curve with coordinates $(u(Q), v(Q)) = (d, e)$, and let $\phi: C \rightarrow C$ be the map defined in terms of the group law on the elliptic curve C , by $\phi(P) = P - Q$. Standard considerations allow us to compute that

$$\begin{aligned} u(\phi(P)) &= m(-P)^2 + am(-P) - d + \frac{v}{u^2}, \\ v(\phi(P)) &= m(-P)(u(\phi(P)) - d) + e, \end{aligned}$$

where

$$m(-P) = \frac{-\frac{v^2}{u^3} - e}{-\frac{v}{u^2} - d}$$

for $P = (u, v)$. Inversion is given by

$$\begin{aligned} u(-P) &= \frac{v^2}{u^2} + a\frac{v}{u} - u = -\frac{v}{u^2}, \\ v(-P) &= \frac{v}{u}u(-P) = -\frac{v^2}{u^3}. \end{aligned}$$

If Q is a point of order 2, applying the inversion formula shows that $e = -d^3$ and $d^3 - ad - 2 = 0$. The universal example E of a subgroup of order 2 in C is defined over

$$S_2 = S[d]/(d^3 - ad - 2),$$

and is generated by the point Q with $(u(Q), v(Q)) = (d, -d^3) = (d, -ad - 2)$.

Let C_2 be the curve with projective equation

$$Y^2Z + (a^2 + 3d - ad^2)XYZ + YZ^2 - X^3 = 0.$$

In the neighborhood of the basepoint, with $u' = X/Y$ and $v' = Z/Y$, this has equation

$$(v')^2 + (a^2 + 3d - ad^2)u'v' + v' = (u')^3.$$

Let C_1 be the curve obtained from C by base change along the evident inclusion $S \rightarrow S_2$. There is an isogeny

$$\psi: C_1 \rightarrow C_2$$

of curves over S_2 , whose kernel is precisely the rank 2 subgroup E . The isogeny can be described easily in terms of (u, v) -coordinates: if $(u(P), v(P)) = (u, v)$ in C_1 and $(u(\psi(P)), v(\psi(P))) = (u', v')$ in C_2 , then we have

$$u' = -u(P)u(\phi(P)), \quad v' = v(P)v(\phi(P)),$$

using the function ϕ defined above, where Q is a point of order 2. (This is a kind of “Lubin isogeny” construction.)

The coordinate u is a uniformizer at the basepoint. Taking the formal expansion of u' in terms of u , we have

$$\begin{aligned} u' &= -du + (ad + 3)u^2 + (-a^2d - 3d^2 - 2a)u^3 + (a^3d + 5ad^2 + 2a^2 + 6d)u^4 \\ &\quad + (-a^4d - 7a^2d^2 - 2a^3 - 16ad - 12)u^5 \\ &\quad + (a^5d + 9a^3d^2 + 2a^4 + 30a^2d + 12d^2 + 32a)u^6 + \cdots, \end{aligned}$$

and the formal expansion of v' in terms of u is

$$\begin{aligned} v' &= +(-ad - 2)u^3 + (2a^2d + 3d^2 + 4a)u^4 + (-3a^3d - 9ad^2 - 6a^2 - 9d)u^5 \\ &\quad + (4a^4d + 18a^2d^2 + 8a^3 + 35ad + 23)u^6 \\ &\quad + (-5a^5d - 30a^3d^2 - 10a^4 - 86a^2d - 27d^2 - 84a)u^7 \\ &\quad + (6a^6d + 45a^4d^2 + 12a^5 + 170a^3d + 126ad^2 + 199a^2 + 63d)u^8 + \cdots. \end{aligned}$$

(The easiest way to derive the equation for the curve C_2 seems to be to calculate the series expansion for u' and v' as above, and then solve for the Weierstrass equation they satisfy.)

Let

$$S_{2,2} = S[d, d']/(d^3 - ad - 2, (d')^3 - (a^2 + 3d - ad^2)d' - 2).$$

This is a pushout in the category of commutative rings of the diagram $S_2 \xleftarrow{s^*} S \xrightarrow{t^*} S_2$, where $s^*: S \rightarrow S_2$ is the usual inclusion, and $t^*: S \rightarrow S_2$ sends $a \mapsto a^2 + 3d - ad^2$. The two ring homomorphisms $e_1^*, e_2^*: S_2 \rightarrow S_{2,2}$ are given by $e_1^*(a) = a, e_1^*(d) = d$ and $e_2^*(a) = a^2 + 3d - ad^2, e_2^*(d) = d'$; they classify the subgroup E_1 of C_1 , and E_2 of C_2 , respectively. The ring $S_{2,2}$ carries the universal example of a nested chain ($E_1 < E_2$) of subgroups of C_1 , with $\text{rank } E_1 = \text{rank } E_2/E_1 = 2$.

Let S_4 be defined as the pullback in the following square of commutative rings.

$$\begin{array}{ccc} S_4 & \longrightarrow & S_{2,2} \\ \downarrow & & \downarrow f^* \\ S & \xrightarrow{s^*} & S_2 \end{array}$$

The map f^* sends $d \mapsto d$ and $d' \mapsto a - d^2$; it classifies the chain of subgroups ($E < C[2]$) in C , where E is the universal example of a rank 2 subgroup, and $C[2]$ is the subgroup of 2-torsion. The universal example of a subgroup of rank 4 of C is defined over the ring S_4 ; the map $S_4 \rightarrow S$ classifies $C[2]$.

4. A MORAVA E -THEORY SPECTRUM

Let $\widehat{S} = \mathbb{Z}_2[[a]]$; this is an extension of the ring S of the previous section. Let \widehat{C} denote the formal completion at the identity of the elliptic curve of the previous section. This defines a formal group over E_0 , which turns out to be a universal deformation for its reduction to $\mathbb{F}_2 = \widehat{S}/(2, a)$, a height 2 formal group. Let E denote the even periodic cohomology theory associated to this formal group over E_0 . This is a Morava E -theory; it is a strictly commutative ring spectrum.

Write $\widehat{S}_2 = \widehat{S} \otimes_{S, s^*} S_2$, and similarly for $\widehat{S}_{2,2}, \widehat{S}_4$, etc. It is a theorem of Strickland [Str98] that

$$\widehat{S}_{p^k} \approx E^0 B\Sigma_{p^k}/(\text{transfer}), \quad k \geq 0.$$

If F is a $K(2)$ -local commutative E -algebra, then $\pi_0 F$ carries the structure of an amplified Γ -ring, as defined above. The “total square”

$$P: \pi_0 F \rightarrow \pi_0 F \otimes_{E_0} E^0 B\Sigma_2/(\text{transfer}) \approx \pi_0 F \otimes_{E_0, s^*} \widehat{S}_2$$

can be recovered by the identification

$$\widehat{S}_2 \approx E^0 B\Sigma_2/(\text{transfer}) \approx \mathbb{Z}_2[[a]][d]/(d^3 - ad - 2),$$

and the formula

$$P(x) = Q_0(x) + Q_1(x)d + Q_2(x)d^2.$$

The cartan formula is read off from the formula $P(xy) = P(x)P(y)$. The commutation relations are obtained from $P(ax) = P(a)P(x)$, using

$$P(a) = a^2 + 3d - ad^2.$$

Since \widehat{S}_2 is a free \widehat{S} -module, there are trace and norm functions $\widehat{S}_2 \rightarrow \widehat{S}$. The operators T and N are defined by $T = \text{trace } P$ and $N = \text{norm } P$.

Let $\text{can}^*: \widehat{S}_2 \rightarrow \widehat{S}/(2) \approx \mathbb{F}_2[[a]]$ be the map defined by $d \mapsto 0$; this classifies the “canonical subgroup”. The operation P has the property that the composite

$$\pi_0 F \rightarrow \pi_0 F \otimes_{\widehat{S}} {}^s \widehat{S}_2 \rightarrow \pi_0 F \otimes_{\widehat{S}} {}^s \widehat{S}/(2) \approx \pi_0 F/(2)$$

is precisely the map sending $x \mapsto x^2$; this is the “frobenius congruence”.

Applying P twice gives a map

$$PP: \pi_0 F \rightarrow \pi_0 F \otimes_{E_0} E^0 B\Sigma_2 / (\text{transfer})_P \otimes_{E_0} E^0 B\Sigma_2 / (\text{transfer}),$$

that is, a map

$$\pi_0 F \rightarrow \pi_0 F \otimes_{\widehat{S}} {}^s \widehat{S}_{2,t^*} \otimes_{\widehat{S}} {}^s \widehat{S}_2 \approx \pi_0 F \otimes_{\widehat{S}} {}^s \widehat{S}_{2,2}.$$

The operation PP factors through $\pi_0 F \otimes_{\widehat{S}} {}^s \widehat{S}_4$, where $\widehat{S}_4 = \lim(\widehat{S}_{2,2} \xrightarrow{f^*} \widehat{S}_2 \xleftarrow{s^*} \widehat{S})$; this is the source of the adem relations. Explicitly, we have

$$\begin{aligned} PP(x) &= P(Q_0 x + d Q_1 x + d^2 Q_2 x) \\ &= P Q_0 x + P d P Q_1 x + P(d^2) P Q_2 x \\ &= P Q_0 x + d' P Q_1 x + (d')^2 P Q_2 x \\ &= Q_0 Q_0 x + d Q_1 Q_0 x + d^2 Q_2 Q_0 x + d' Q_0 Q_1 x + d d' Q_1 Q_1 x + d^2 d' Q_2 Q_1 x \\ &\quad + (d')^2 Q_0 Q_2 x + d(d')^2 Q_1 Q_2 x + d^2 (d')^2 Q_2 Q_2 x \end{aligned}$$

in $\pi_0 F \otimes_{E_0} {}^s \widehat{S}_{2,2}$. Observe that $P(d) = d'$ here, since $P: \widehat{S}_2 \rightarrow \widehat{S}_{2,2}$ corresponds to e_2^* . The projection of the above formula under $f^*: \widehat{S}_{2,2} \rightarrow \widehat{S}_2$ is

$$\begin{aligned} &Q_0 Q_0 x + a Q_0 Q_1 x - 2 Q_1 Q_1 x + a^2 Q_0 Q_2 x - 2a Q_1 Q_2 x + 4 Q_2 Q_2 x \\ &+ (Q_1 Q_0 x - 2 Q_2 Q_1 x + 2 Q_0 Q_2 x)d + (Q_2 Q_0 x - Q_0 Q_1 x - a Q_0 Q_2 x + 2 Q_1 Q_2 x)d^2. \end{aligned}$$

The adem relations are obtained by setting the coefficients of d and d^2 to 0. The operation Ψ is what's left over; i.e., it arises from the homomorphism $\widehat{S}_4 \rightarrow \widehat{S}$.

Let Γ' denote the algebra of power operations for E as described in [Rez]. It has a direct sum decomposition $\Gamma' \approx \bigoplus \Gamma'[k]$, where the pieces come from the E -homology of $B\Sigma_{p^k}$. Let $\Gamma_{\widehat{S}} = \widehat{S} \otimes_R \Gamma$. There is a degree preserving ring homomorphism $\alpha: \Gamma_{\widehat{S}} \rightarrow \Gamma'$, which is an isomorphism in degrees 0 and 1. It is easy to see that Γ' is generated by $\Gamma'[1]$ using a simple transfer argument, and thus α is surjective. We conclude that α is an isomorphism for degree reasons, using the rank calculations of [ST97].

We have $E^0 \mathbb{C}P^\infty \approx \widehat{S}[[u]]$; information about the action of Γ on this ring can be read off from the formal expansion of u' in terms of u given in §3. Thus

$$\begin{aligned} Q_0(u) &= -3u^2 - 2au^3 + 2a^2u^4 + (-2a^3 - 12)u^5 + (2a^4 + 32a)u^6 + \cdots, \\ Q_1(u) &= -u + au^2 - a^2u^3 + (a^3 + 6)u^4 + (-a^4 - 16a)u^5 + (a^5 + 30a^2)u^6 + \cdots, \\ Q_2(u) &= -3u^3 + 5au^4 - 7a^2u^5 + (9a^3 + 12)u^6 + \cdots. \end{aligned}$$

The action of Γ on $E^0 S^2$ is easily read off from this; the Γ -module ω is the kernel of $E^0 S^2 \rightarrow E^0$.

It is not surprising that the apparatus of (2-primary) power operations can actually be defined over the ring $S = \mathbb{Z}[a, D^{-1}]$, since all the structure flows from the existence of the

elliptic curve C over this ring. It *is* a bit surprising that all this apparatus appears to be defined over the ring $R = \mathbb{Z}[a]$, where the curve can fail to be smooth. However, it appears that all the formulas lift to this setting, and this is the ground ring I chose to use for the presentation of §2.

REFERENCES

- [BW05] James Borger and Ben Wieland, *Plethystic algebra*, Adv. Math. **194** (2005), no. 2, 246–283. MR MR2139914 (2006i:13044)
- [GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at <http://www.math.uiuc.edu/Macaulay2/>.
- [Pri70] Stewart B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60. MR MR0265437 (42 #346)
- [Rez] Charles Rezk, *The congruence criterion for power operations in Morava E -theory*, in preparation.
- [ST97] Neil P. Strickland and Paul R. Turner, *Rational Morava E -theory and DS^0* , Topology **36** (1997), no. 1, 137–151. MR 97g:55005
- [Str98] N. P. Strickland, *Morava E -theory of symmetric groups*, Topology **37** (1998), no. 4, 757–779. MR 99e:55008

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